Modified version of a two point gradient Levenberg-Marquardt method for nonlinear ill-posed operator equations

Dissertation submitted to
Visvesvaraya National Institute of Technology, Nagpur
in partial fulfillment of the requirement for the award of
the degree

Master of Science

in

Mathematics

by

Puja Parmar
(MS23MTH018)

under the guidance of

Dr. Pallavi Mahale

Department of Mathematics
Visvesvaraya National Institute of Technology
Nagpur, 440010 (India)
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Declaration

I, Puja Parmar, hereby declare that this dissertation titled **Modified version of a two point gradient Levenberg-Marquardt method for nonlinear ill-posed operator equations** is carried out by me in the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted earlier whole or in part for the award of any degree/diploma at this or any other Institution/University.

Puja Parmar
Date:

Certificate

This is to certify that the dissertation titled **"Modified version of a two-point gradient Levenberg-Marquardt method for nonlinear ill-posed operator equations"** is submitted by **Puja Parmar** in partial fulfillment of the requirements for the award of the degree of **Master of Science in Mathematics**, VNIT Nagpur. The work is comprehensive, complete and fit for final evaluation.

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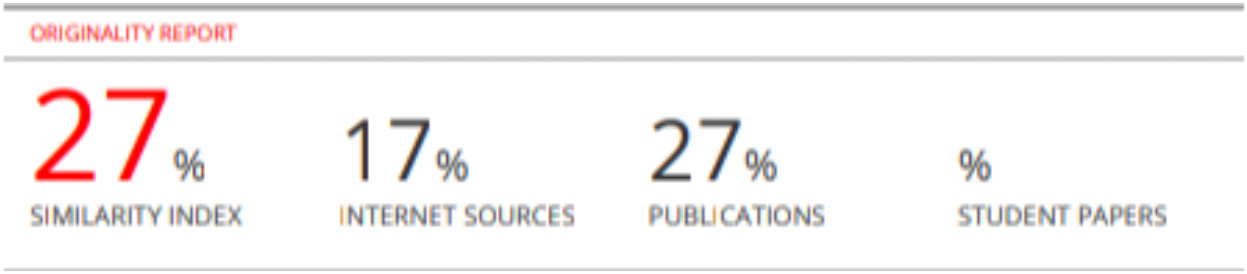

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Puja Parmar

ABSTRACT

In this dissertation, we have studied convergence analysis of modified version of a two-point gradient Levenberg-Marquardt method considered by G.Gao et. al., in 2022. Techniques which are used for convergence analysis in the dissertation is motivated by one used by Gao et.al. The advantage of the modified form the method is that one need not calculate Fréchet derivative at each iteration step. Convergence analysis in this dissertation is done under Morozov's type stopping rule.

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NOTATIONS

We use the following standard notations and symbols without defining them explicitly:
$\subseteq$: subset of
$\in$: belongs to
$\mathbb{N}$: The set of all natural numbers.
$\forall$: for all
$\exists$: there exists or there exist
$\Rightarrow$: implies or imply
$\Leftrightarrow$: if and only if
$\rightharpoonup$: weakly convergent
$<,>$: Inner product.
$\|.\|$: Norm.
$D(\mathcal{F})$: domain of an operator $\mathcal{F}$
$C[a,b]$: set of continuous function on [a,b]
$L[a,b]$: set of integrable function on [a,b]

CHAPTER-1

Introduction

1 Introduction

1.1 Ill-posed operator equation

Definition: Operator equation of the form

$$\mathcal{F}(x) = y, \tag{1.1}$$

where $\mathcal{F} : X \to Y$ is a operator between Hilbert spaces X and Y, is said to be well-posed operator equation if for all $y \in Y$, it satisfy following properties:

- $\mathcal{F} : D(\mathcal{F}) \subset X \to Y$ is bijective.
- $\mathcal{F}^{-1} : R(\mathcal{F}) \subset Y \to X$ exists and continuous.

More specifically an operator equation is said to be well-posed if

- it has a solution,
- it has at most one solution, i.e., for every $y \in Y$ there exist a unique $x \in X$ which satisfy (1.1)
- the solution depends continuously on the data, i.e. for every $y \in Y$ and every for $\epsilon > 0$ there exist $\delta > 0$ such that if $\tilde{y} \in Y$ with $\|y - \tilde{y}\| \leq \delta$, and $x, \tilde{x} \in D(\mathcal{F})$ are such that $\mathcal{F}x = y$ and $\mathcal{F}\tilde{x} = \tilde{y}$ then $\|x - \tilde{x}\| < \epsilon$.

If one of the above conditions does not satisfies then the operator equation (1.1) is called ill-posed operator equation.

1.2 Examples of ill-posed operator equation

1.2.1 Linear ill-posed operator equation:

Compact operator equation: Let $\mathcal{F} : X \to Y$ be a compact operator of infinite rank between two Hilbert spaces X and Y. Then for all $x \in X$, making use of the singular value representation, $\mathcal{F}$ can be expressed as

$$\mathcal{F}(x) = \sum_{n=1}^{\infty} \mu_n \langle x, u_n \rangle v_n,$$

where $\{u_n : n \in \mathbb{N}\}$ and $\{v_n : n \in \mathbb{N}\}$ are orthonormal bases of $\mathcal{N}(T)^{\perp}$ and $\mathcal{R}(T)$, respectively, and $\{\mu_n\}$ is a decreasing sequence of non-negative numbers converging to 0.

For $x \in X$ and $y \in Y$, if

$$\mathcal{F}(x) = y,$$

then

$$\langle x, u_n \rangle = \frac{\langle y, v_n \rangle}{\mu_n}, \quad \text{for all } n \in \mathbb{N}.$$

To have a solution from this equation, we need

$$\sum_{n=1}^{\infty} \frac{|\langle y, v_n \rangle|^2}{\mu_n^2} < \infty,$$

and also

$$\|\mathcal{F}u_n\| = \mu_n \to 0 \quad \text{as } n \to \infty,$$

while

$$\|u_n\| = 1 \quad \text{for all } n \in \mathbb{N}.$$

So, here $\mathcal{F}$ is not bounded below. Therefore, it is ill-posed by definition. To support this argument, we can also make use of the known fact that $\mathcal{R}(\mathcal{F})$ is not closed as the domain $D(\mathcal{F})$ is of infinite dimension.

In general, if $x \in X$ and $x_n = x + \frac{u_n}{\sqrt{\mu_n}}$ for $n \in \mathbb{N}$, then

$$\mathcal{F}(x_n) = \mathcal{F}(x) + \sqrt{\mu_n} v_n.$$

We obtain

$$\|\mathcal{F}(x_n) - \mathcal{F}(x)\| = \sqrt{\mu_n} \to 0,$$

whereas

$$\|x_n - x\| = \frac{1}{\sqrt{\mu_n}} \to \infty \quad \text{as } n \to \infty,$$

which implies that due to a small perturbation y^δ in the data y, a substantial change occurs in the solution x. Therefore, the equation is ill-posed.

1.2.2 Non linear ill-posed operator equation:

Hammerstien integral equation: We know that the Hammerstein integral equation is of the form

$$\mathcal{F}(z)(t) = \int_0^1 c(t,s)\, g(s, z(s))\, ds, \quad s, t \in [0,1],$$

where $\mathcal{F} : D(\mathcal{F}) \subseteq L^2[0,1] \to L^2[0,1]$, and $c(\cdot,\cdot) \in L^2([0,1] \times [0,1])$ is a non-degenerate kernel of the integral equation, satisfying the condition

$$\int_0^1 \int_0^1 |c(t,s)|^2\, dt\, ds \leq \infty.$$

and also the function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the condition

$$|g(s,t)| \leq k|t| + q(s), \quad \text{with } k > 0 \text{ and } q \in L^2[0,1].$$

Here, the kernel $c(t,s)$ is given by

$$c(t,s) = \begin{cases} 0, & \text{if } 0 \leq t \leq s \leq 1, \\ t - s, & \text{if } 0 \leq s < t \leq 1, \end{cases}$$

and the function $g(s,t) = t^3$. Therefore, in this case, the integral equation becomes

$$\mathcal{F}(z)(t) = \int_0^t (t-s) z^3(s)\, ds.$$

It is easy to prove that $\mathcal{F}$ is continuous, compact, injective, and weakly closed. By a weakly closed operator $\mathcal{F}$, we mean that for any sequence $\{x_n\} \in D(\mathcal{F})$, the weak convergence $x_n \rightharpoonup x \in X$ and $\mathcal{F}(x_n) \rightharpoonup y$ in Y implies that $x \in D(\mathcal{F})$ and $\mathcal{F}(x) = y$ This shows that $\mathcal{F}^{-1}$ is not continuous, and hence the problem becomes ill-posed.

1.3 Notation, Preliminaries, and Definition

In this section, we will give some definitions and results which will be used in the dissertation.

Definition 1.1. ([10]) Let $\mathcal{F} : X \to Y$ be operator between Hilbert spaces X and Y . Then

$$N(\mathcal{F}) = \{x \in X : \mathcal{F}x = 0\},$$

is called null space of $\mathcal{F}$ and

$$R(\mathcal{F}) = \{\mathcal{F}x : x \in X\}$$

is called range space of $\mathcal{F}$

Definition 1.2. ([10]) Let $\mathcal{F} : X \to Y$ be a operator between inner product spaces X and Y. The operator $\mathcal{F}$ is said to have adjoint if there exists a linear operator $\mathcal{F}^* : Y \to X$ such that $\langle \mathcal{F}x, y\rangle = \langle x, \mathcal{F}^*y\rangle \quad \forall x \in X,\ \forall y \in Y$. Here $\mathcal{F}^*$ is known as adjoint of $\mathcal{F}$.

Definition 1.3. ([10]) A bounded linear operator $\mathcal{F} : X \to X$ is said to be a self adjoint operator if $\mathcal{F}^* = \mathcal{F}$.

Definition 1.4. Fréchet Derivative:
Let X and Y be Banach spaces and $D(\mathcal{F}) \subseteq X$ be an open subset of X. A function $\mathcal{F} : D(\mathcal{F}) \subseteq X \to Y$ is said to be Fréchet differentiable at $x \in D(\mathcal{F})$ if x is a interior point of $D(\mathcal{F})$ and if there exist a bounded linear operator $H : X \to Y$ such that

$$\lim_{\|\varrho\|\to 0} \frac{\|\mathcal{F}(x+\varrho) - \mathcal{F}(x) - H\varrho\|}{\|\varrho\|} = 0$$

and in this case, the operator H is called the Fréchet derivative of $\mathcal{F}$ at x and it is denoted by $\mathcal{F}'(x)$.

Definition 1.5. ([10]) An operator $\mathcal{F} : X \to Y$ is said to be completely continuous if every bounded sequence $\{x_n\} \subset X$,

$$x_n \rightharpoonup x \Rightarrow \mathcal{F}(x_n) \to \mathcal{F}(x).$$

Definition 1.6. ([10]) Let X be a normed linear space. A sequence $\{x_n\} \subset X$ is said to converge **weakly** to an element $x \in X$ if:

$$\lim_{n\to\infty} f(x_n) = f(x) \quad \text{for all } f \in X^*,$$

where X^* denotes the dual space of X, that is, the space of all bounded linear functionals on X. This is denoted by, $x_n \rightharpoonup x$.

1.4 Regularization

If the operator equation (1.1) is ill-posed due to the instability of its solution, a stable approximate solution can be obtained through regularization methods. These methods involve replacing the ill-posed problem (1.1) with a family of well-posed problems that depend on specific parameters. The resulting solution from this approach is referred to as the regularized solution. let us assume that (1.1) has a unique solution namely x^* and because of practical challenges, instead of exact data y we have consider perturbed data y^δ satisfying

$$\|y - y^\delta\| \leq \delta. \tag{1.2}$$

where $\delta > 0$ is small noise level.

There are essentially two main classes of regularization methods:

1. **Continuous Regularization Methods**
2. **Discrete or Iterative Regularization Methods**

1.4.1 Continuous regularization method

In continuous regularization methods the regularization parameter is a continuous variable which has to be chosen appropriately depending on the noisy data. Tikhonov regularization is a widely used method for solving ill-posed operator equations. In this method, the regularized solution x_n^δ is obtained as a solution of the following minimization problem

$$\min_{v \in D(\mathcal{F})} J_{\beta,\delta}(v)$$

where

$$J_{\beta,\delta}(v) := \|\mathcal{F}(v) - y^\delta\|^2 + \beta\|v - x_0\|^2, \quad v \in D(\mathcal{F}).$$

Here, $\beta > 0$ is a regularization parameter and $x_0 \in D(\mathcal{F})$ is an initial approximation of $x^\dagger$.

1.4.2 Iterative regularization methods

An iterative method with iterations defined by

$$x_{i+1}^\delta = \Phi\left(x_0^\delta, x_1^\delta, \ldots, x_i^\delta; y^\delta\right), \quad x_0^\delta := x_0$$

for a known function together with a stopping rule which determines a stopping index $n_* \in N$ is called an iterative regularization method if

$$x_{n_\delta}^\delta \to x^* \quad \text{as} \quad \delta \to 0.$$

Here, $x_0 \in D(\mathcal{F})$ is a known initial approximation of the exact solution x^*. Landweber iterative method ([5] [6]), Gauss Newton iterative method ([9] [11]) and Levenberg Marquardt method ([5] [8]) are examples of iterative type method.

1.5 Levenberg-Marquardt Method

Levenberg-Marquardt iterative method was first introduced by Martin Hanke in [5], in the Hilbert space framework, in which they minimizing the functional

$$\|y^\delta - \mathcal{F}(x_n^\delta) - \mathcal{F}'(x_n^\delta)h\|^2 + \beta_n\|h\|^2, \tag{1.3}$$

for $h = h_n = x^\dagger - x_n^\delta$. Here β_n is a suitable regularization parameter and $\mathcal{F}'(x_n^\delta)$ is Fréchet derivative. Then the next iterate has been define as

$$x_{n+1}^\delta = x_n^\delta + h_n.$$

In [5] Hanke has choosen parameter regularization as a solution of the following equation

$$\|y^\delta - \mathcal{F}(x_n^\delta) - \mathcal{F}'(x_n^\delta)(x_{n+1}^\delta(\beta) - x_n^\delta)\| = \sigma\|y^\delta - \mathcal{F}(x_n^\delta)\|$$

for some fixed $\sigma \in (0, 1)$. Here $x_{n+1}^\delta(\beta)$ denotes minimization of (1.3). To avoid the minimization of the functional define in the method (1.3), jin. Qinian in [8], has considered following explicit form of the LM method, where iteration have been defined as

$$x_{n+1}^\delta = x_n^\delta - (\beta_n^\delta I + \mathcal{F}'(x_n^\delta)^*\mathcal{F}(x_n^\delta))^{-1}\mathcal{F}'(x_n^\delta)^*(\mathcal{F}(x_n^\delta) - y^\delta), \tag{1.4}$$

where β_n is regularization parameter and I be the identity parameter.

1.6 Stopping rule

The selection of a stopping index is a critical component of iterative regularization methods, as continuing iterations beyond a certain point can lead to an uncontrolled amplification of propagated data errors. One of the most well-known criteria for determining this stopping point is the Morozov discrepancy principle. According to this principle, the stopping index n_* is chosen as the smallest index such that the following condition holds:

$$\|\mathcal{F}(x_{n_*}^\delta) - y^\delta\| \leq \tau\delta \leq \|\mathcal{F}(x_n^\delta) - y^\delta\|, \quad n = 0, 1, \ldots, n_* - 1, \tag{1.5}$$

for an appropriate constant $\tau \geq 1$.

1.7 Drawback of Levenberg-Marquardt method

The numerical results in [8] required many iterations to meet the stopping rule (1.3), which in practical applications often involves solving a large linear system involving the operator $(\mathcal{F}'(x)^*\mathcal{F}'(x) + \beta_n^\delta I)^{-1}$ at each iteration. Since this is often computationally expensive, it is highly desirable to employ acceleration strategies to improve computational efficiency and enhance practical applicability. In [7], Nesterov first introduced an acceleration technique for nonlinear convex optimization, where two iterates x_n^δ and x_{n-1}^δ are combined and obtain

$$z_n^\delta = x_n^\delta + \frac{n-1}{n-\omega-1}(x_n^\delta - x_{n-1}^\delta)$$

to achieve faster convergence, where $\omega \geq 3$ be a constant. In several studies, the term $\frac{n-1}{n-\omega-1}$ is replaced by a combination parameter κ_n^δ leading to the formulation of a two-point gradient method.

1.8 A two-point gradient Levenberg-Marquardt method

To over come this issue and speed up the convergence of Levenberg-Marquardt method (1.4) and to improve the efficiency of solving inverse problems, Gao et. al., in [3] considered the following method

$$z_n^\delta = x_n^\delta + \kappa_n^\delta(x_n^\delta - x_{n-1}^\delta)$$

where $\kappa_n^\delta \geq 0$ be the combination parameter and $\beta_n^\delta > 0$ chosen such that

$$\beta_n^\delta\|(\mathcal{F}'(z_n^\delta)\mathcal{F}'(z_n^\delta)^* + \beta_n^\delta I)^{-1}(\mathcal{F}(z_n^\delta) - y^\delta)\| = \sigma\|\mathcal{F}(z_n^\delta) - y^\delta)\|.$$

Thus

$$x_{n+1}^\delta = z_n^\delta - \sigma_n^\delta\mathcal{F}'(z_n^\delta)^*\left(\mathcal{F}'(z_n^\delta)\mathcal{F}'(z_n^\delta)^* + \beta_n^\delta I\right)^{-1}\left(\mathcal{F}(z_n^\delta) - y^\delta\right),$$

where $\sigma_n^\delta > 0$ be the adaptive stepsize. To stop the iteration, they apply Morozov type stopping criteria which is defined in (1.5). Then used $x_{n_*}^\delta$ as an approximate solution. In this dessertation, we will study modified form of the above method.

1.9 Outline

In chapter 1 :- We have given some example of linear and nonlinear ill-posed operator equations. We have also discussed some definition which will be used throughout the dissertation. Then we have summarise the basic concepts related to regularization of an ill-posed operator

equations.

In chapter 2 :- We will formulate the modified form of the method considered in [3]. Using similar techniques as used in [3], we show method is well defined and stopping index exists and finite. We will study convergence of the method in case of exact data (i.e $\delta = 0$). We will also show the method is asymptotically stable. In end of the chapter, we give convergence of method in case of data with error.

CHAPTER-2

2 Modified version of a two point gradient Levenberg-Marquardt method for nonlinear ill-posed operator equations

Motivated by the work done by Gao et. al, in [3], we define modified two-point gradient Levenberg-Marquardt method which require calculation of Fréchet derivative at an initial guess x_0. The method is as follow:
Let $x_0 = z_0 \in X$, then we define

$$z_n^\delta = x_n^\delta + \kappa_n^\delta (x_n^\delta - x_{n-1}^\delta)$$

$$x_{n+1}^\delta = z_n^\delta - \sigma_n^\delta \mathcal{F}'(z_0^\delta)^* \left(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I\right)^{-1} \left(\mathcal{F}(z_n^\delta) - y^\delta\right), \tag{2.1}$$

where $\kappa_n^\delta \geq 0$ be the combination parameters, $\sigma_n^\delta > 0$ be the stepsize and β_n^δ chosen to satisfy,

$$\beta_n^\delta \|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I)^{-1}(\mathcal{F}(z_n^\delta) - y^\delta)\| = \sigma \|\mathcal{F}(z_n^\delta) - y^\delta)\|, \tag{2.2}$$

with a positive constant $0 < \sigma < 1$.

2.1 Basic assumptions and some preliminaries results

In this section, we present some assumptions and praliminary results related to the modified method.

Assumption 2.1. Let $\bar{\mathcal{B}}_\rho(x_0) = \{x \in X : \|x - x_0\| \leq \rho\}$ denote a closed ball which is centred at x_0 and of the radius ρ. We assume the following

1. The operator $\mathcal{F}$ is completely continuous and there exist a solution for (1.1).
2. There exist a constant $\mathcal{L} > 0$ such that $\|\mathcal{F}'(x)\| \leq \mathcal{L}$, for all $x \in \bar{\mathcal{B}}_\rho(x_0)$. ,
3. The Banach space Z such that $\bigcup_{x\in X} \mathbb{R}(\mathcal{F}'(x)^*) \subset Z$ with $Z \subset X$ compactly embedded in X. Furthermore, there exists a constant $\hat{\mathcal{L}} > 0$ such that $\|\mathcal{F}'(x)^*\| \leq \hat{\mathcal{L}}$ for all $x \in \bar{\mathcal{B}}_\rho(x_0)$.
4. For every x, $\bar{x} \in \bar{\mathcal{B}}_\rho(x_0)$, there exist $\gamma \in (0, 1)$ such that

$$\|\mathcal{F}(x) - \mathcal{F}(\tilde{x}) - \mathcal{F}'(x_0)(x - \tilde{x})\| \leq \gamma \|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\|. \tag{2.3}$$

Proposition 2.2. Let Assumption 2.1 be hold. Then

(a) $(1-\gamma)\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\| \leq \|\mathcal{F}'(x_0)(x - \tilde{x})\| \leq (1+\gamma)\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\|, \quad \forall x, \tilde{x} \in \overline{\mathcal{B}}_\rho(x_0)$

(b) $\|\mathcal{F}'(x_0)(x^* - \tilde{x})\| \leq 2(1+\gamma)\|\mathcal{F}(x) - y\| + (1+\gamma)\|\mathcal{F}(\tilde{x}) - y\|, \forall x, \tilde{x} \in \overline{\mathcal{B}}_\rho(x_0)$

(c) If $x^* \in \bar{\mathcal{B}}_\rho(x_0)$ such that $\mathcal{F}'(x^*) = y$. Then there exists a constant $\tilde{\mathcal{L}} > 0$ such that

$$\|\mathcal{F}'(x_0)\mathcal{F}'(x_0)^*\| \geq \tilde{\mathcal{L}}. \tag{2.4}$$

Proof. First we proof left hand side of inequality (a), using equation (2.3), we get

$$\|\mathcal{F}(x) - \mathcal{F}(\tilde{x}) - \mathcal{F}'(x_0)(x - \tilde{x})\| \leq \gamma\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\|,$$

we can writes

$$\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\| - \|\mathcal{F}'(x_0)(x - \tilde{x})\| \leq \gamma\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\|,$$

Thus, we get

$$(1 - \gamma)\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\| \leq \|\mathcal{F}'(x_0)(x - \tilde{x})\|. \tag{2.5}$$

for all $x, \tilde{x} \in \bar{\mathcal{B}}_\rho(x_0)$.

Next, we prove the right-hand side of inequality (a). Using equation (2.3), we get

$$\|\mathcal{F}'(x)(x - \tilde{x}) - (\mathcal{F}(x) - \mathcal{F}(\tilde{x}))\| \leq \gamma\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\|, \forall x, \tilde{x} \in \bar{\mathcal{B}}_\rho(x_0).$$

so, we have

$$\|\mathcal{F}'(x)(x - \tilde{x})\| - \|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\| \leq \gamma\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\|.$$

Hence,

$$\|\mathcal{F}'(x)(x - \tilde{x})\| \leq (1 + \gamma)\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\|. \tag{2.6}$$

From (2.5)and (2.6), we obtain

$$(1 - \gamma)\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\| \leq \|\mathcal{F}'(x)(x - \tilde{x})\| \leq (1 + \gamma)\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\|.$$

Hence, the first assertion (a) is proved.

Now, we prove inequality (b). Using inequality (a), we note that

$$\begin{aligned}
\|\mathcal{F}'(x^*)(x - \tilde{x})\| &\leq (1 + \gamma)\|\mathcal{F}(x^*) - \mathcal{F}(\tilde{x})\| \\
&= (1 + \gamma)\|y - \mathcal{F}(x) + \mathcal{F}(x) - \mathcal{F}(\tilde{x})\| \\
&\leq (1 + \gamma)(\|y - \mathcal{F}(x)\| + \|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\|) \\
&\leq (1 + \gamma)\|y - \mathcal{F}(x)\| + (1 + \gamma)\|\mathcal{F}(x) - y + y - \mathcal{F}(\tilde{x})\| \\
&\leq (1 + \gamma)\|y - \mathcal{F}(x)\| + (1 + \gamma)\|\mathcal{F}(x) - y\| + (1 + \gamma)\|y - \mathcal{F}(\tilde{x})\| \\
&= 2(1 + \gamma)\|y - \mathcal{F}(x)\| + (1 + \gamma)\|y - \mathcal{F}(\tilde{x})\|.
\end{aligned}$$

Hence,

$$\|\mathcal{F}'(x^*)(x^* - \tilde{x})\| \leq 2(1 + \gamma)\|\mathcal{F}(x) - y\| + (1 + \gamma)\|\mathcal{F}(\tilde{x}) - y\|.$$

Now we prove the third condition by contradiction.

According to the tangential cone condition (2.3), if $\mathcal{F}'(x) = 0$ for some $x \in \bar{\mathcal{B}}_\rho(x_0)$, then:

$$\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\| \leq \gamma\|\mathcal{F}(x) - \mathcal{F}(\tilde{x})\|, \quad \forall x, \tilde{x} \in \bar{\mathcal{B}}_\rho(x_0),$$

where $\gamma \in (0, 1)$, which is a contradiction. So $\exists\tilde{\mathcal{L}} > 0$ such that $|\mathcal{F}'(x_0)\mathcal{F}'(x_0)^*| \geq 0$.

Alternatively, assume there exists a sequence $\{x_k\} \subset \bar{\mathcal{B}}_\rho(x_0)$ such that:

$$\lim_{k\to\infty} \|\mathcal{F}'(x_k)\| = 0.$$

Then,

$$\lim_{k\to\infty} \|\mathcal{F}(x_k) - \mathcal{F}(\tilde{x}) - \mathcal{F}'(x_k)(x_k - \tilde{x})\| \leq \gamma \lim_{k\to\infty} \|\mathcal{F}(x_k) - \mathcal{F}(\tilde{x})\|, \quad \forall\tilde{x} \in \bar{\mathcal{B}}_\rho(x_0).$$

Moreover,

$$\begin{aligned}
\|\mathcal{F}(x_k) - \mathcal{F}(\tilde{x})\| &\leq \|\mathcal{F}(x_k) - \mathcal{F}(\tilde{x}) - \mathcal{F}'(x_k)(x_k - \tilde{x})\| + \|\mathcal{F}'(x_k)(x_k - \tilde{x})\| \\
&\leq \gamma\|\mathcal{F}(x_k) - \mathcal{F}(\tilde{x})\| + \|\mathcal{F}'(x_k)(x_k - \tilde{x})\|.
\end{aligned}$$

Taking the limit as $k \to \infty$, we get:

$$\lim_{k\to\infty} \|\mathcal{F}(x_k) - \mathcal{F}(\tilde{x})\| \leq \gamma \lim_{k\to\infty} \|\mathcal{F}(x_k) - \mathcal{F}(\tilde{x})\|.$$

Since $\gamma \in (0, 1)$, this leads to a contradiction, which violates the continuity of the operator. Hence, we conclude that equation (2.4) is valid.

□

2.2 Method for noisy data

In this section, we define following assuption. Let

$$r_n^\delta := \mathcal{F}(z_n^\delta) - y^\delta,$$

$$\begin{aligned} \nu_n^\delta &:= (\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I)^{-1} r_n^\delta, \\ u_n^\delta &:= \mathcal{F}'(z_0^\delta)^* \nu_n^\delta. \end{aligned} \tag{2.7}$$

Given a number $\rho > 0$ and a fixed initial point x_0. Let

$$\mathcal{S}(x_0, \rho) := \left\{ x \in \overline{\mathcal{B}}_\rho(x_0) \subset X \ : \ \mathcal{F}(x) = y \right\}$$

define solution of (1.1) Let $x_0 = z_0 = x_0^\delta = z_0^\delta \in X$ be initial guess and κ_n^δ be an combination parameter satisfying following:

$$\kappa_0^\delta = 0, \quad 0 \le \kappa_n^\delta \le 1, \quad \forall n \in \mathbb{N}. \tag{2.8}$$

with $\delta \ge 0$. Furthermore, we assume that positive constants $\sigma_0, \gamma, \tau, \sigma, M$ are choose such that

$$\gamma < \sigma < 1, \quad \tau > \frac{1+\gamma}{\sigma - \gamma}, \quad \sigma_0 > 0, \quad \text{and} \quad 0 < M < \frac{\mathcal{L}(1-\gamma)}{\hat{\mathcal{L}}}. \tag{2.9}$$

We define

$$z_n^\delta = x_n^\delta + \kappa_n^\delta (x_n^\delta - x_{n-1}^\delta). \tag{2.10}$$

Let $\beta_n^\delta > 0$ be such that

$$\beta_n^\delta \|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I)^{-1}(\mathcal{F}(z_n^\delta) - y^\delta)\| = \sigma \|\mathcal{F}(z_n^\delta) - y^\delta)\|, \tag{2.11}$$

and

$$\sigma_n^\delta := \min\left\{ \frac{\sigma_0 \langle \nu_n^\delta, r_n^\delta \rangle}{\|u_n^\delta\|^2}, M \right\}, \tag{2.12}$$

and let the iteration be define as

$$x_{n+1}^\delta = z_n^\delta - \sigma_n^\delta \mathcal{F}'(z_0^\delta)^* \left(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I \right)^{-1} \left(\mathcal{F}(z_n^\delta) - y^\delta \right) \tag{2.13}$$

To stop the iteration, we use Morozov type stopping criteria given in (1.5). Then use $x_{n_*}^\delta$ as an approximate solution.

2.3 Some Lemmas and Theorems

Lemma 2.3. Let Assumption 2.1 be hold. Then the regularization parameter β_n^δ satisfying (2.11) is bounded, i.e. there exist $J_1, J_2 > 0$ such that

$$J_1 \le \beta_n^\delta \le J_2, \tag{2.14}$$

where $J_2 := \frac{\sigma}{1-\sigma}\mathcal{L}\hat{\mathcal{L}}$, $J_1 := \sigma\tilde{\mathcal{L}}$.

Proof. Using (2.11), we know that $\beta_n^\delta > 0$ and

$$\begin{aligned} \sigma\|\mathcal{F}(z_n^\delta) - y^\delta)\| &= \beta_n^\delta \|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I)^{-1}(\mathcal{F}(z_n^\delta) - y^\delta)\|, \\ &= \beta_n^\delta \left\| \frac{(\mathcal{F}(z_n^\delta) - y^\delta)}{\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I} \right\|, \\ &\ge \beta_n^\delta \frac{\|(\mathcal{F}(z_n^\delta) - y^\delta)\|}{\|\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^*\| + \beta_n^\delta}, \\ &\ge \beta_n^\delta \frac{1}{\mathcal{L}\hat{\mathcal{L}} + \beta_n^\delta} \|\mathcal{F}(z_n^\delta) - y^\delta\|, \end{aligned}$$

Then, we have

$$\sigma \geq \beta_n^\delta \frac{1}{\mathcal{L}\hat{\mathcal{L}} + \beta_n^\delta},$$

which implies $\beta_n^\delta \leq \frac{\sigma}{1-\sigma}\mathcal{L}\hat{\mathcal{L}} = J_2$. On the other hand,

$$\begin{aligned} \sigma\|\mathcal{F}(z_n^\delta) - y^\delta)\| &= \beta_n^\delta \|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I)^{-1}(\mathcal{F}(z_n^\delta) - y^\delta)\|, \\ &\leq \beta_n^\delta \frac{\|\mathcal{F}(z_n^\delta) - y^\delta\|}{\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I\|} \end{aligned}$$

Since $\beta_n^\delta > 0$, then

$$\beta_n^\delta \geq \sigma\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I\| \geq \sigma\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^*\| \geq \sigma\tilde{\mathcal{L}} = J_1$$

Therefore the proof is complete. □

In the below result we show existence and uniqueness of parameter β_n^δ which satisfy (2.11)

Lemma 2.4. Suppose Assumption 2.1 and equation (2.9) hold for every $0 \leq n \leq n_* - 1$, then parameter β_n^δ satisfying (2.11) exists and unique.

Proof. Denote

$$Q_n(\beta) := \beta\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-1}(\mathcal{F}(z_n^\delta) - y^\delta)\|$$

Note that $Q_n(\beta)$ is continuous and $Q_n(\beta)$ tends to $\|\mathcal{F}(z_n^\delta) - y^\delta\|$, as β approaches infinity. Now, combining equation (2.3) and (2.1), we have

$$\begin{aligned} Q_n(\beta) &= \beta\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-1}(\mathcal{F}'(z_0^\delta) - y^\delta + y - y \\ &\quad +\mathcal{F}'(z_0^\delta)(x^* - z_n^\delta) - \mathcal{F}'(z_0^\delta)(x^* - z_n^\delta)\| \\ &\leq \beta\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-1}\mathcal{F}'(z_0^\delta)(x^* - z_n^\delta)\| + \|y^\delta - y\| \\ &\quad +\beta\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-1}(y - \mathcal{F}(z_n^\delta) - \mathcal{F}'(z_0^\delta)(x^* - z_n^\delta))\| \\ &\leq \beta\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-1}\mathcal{F}'(z_0^\delta)(x^* - z_n^\delta)\| \\ &\quad +\|y^\delta - y\| + \gamma\|\mathcal{F}(x^*) - \mathcal{F}(z_n^\delta)\| \\ &\leq \beta\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-1}\mathcal{F}'(z_0^\delta)(x^* - z_n^\delta)\| \\ &\quad +\|y^\delta - y\| + \gamma\|y - \mathcal{F}(z_n^\delta) - y^\delta + y^\delta)\| \\ &\leq \beta\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-1}\mathcal{F}'(z_0^\delta)^*(x^* - z_n^\delta)\| \\ &\quad +\|y^\delta - y\| + \gamma\|y^\delta - \mathcal{F}(z_n^\delta)\| + \gamma\|y^\delta + y\| \\ &\leq \beta\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-1}\mathcal{F}'(z_0^\delta)(x^* - z_n^\delta)\| \\ &\quad +\gamma\|\mathcal{F}(z_n^\delta) - y^\delta\| + (\gamma+1)\|y^\delta - y\| \\ &\leq \beta\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-1}\mathcal{F}'(z_0^\delta)(x^* - z_n^\delta)\| \\ &\quad +\gamma\|\mathcal{F}(z_n^\delta) - y^\delta\| + (\gamma+1)\frac{\|\mathcal{F}(z_n^\delta) - y^\delta\|}{\tau} \\ &\leq \beta\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-1}\mathcal{F}'(z_0^\delta)(x^* - z_n^\delta)\| + \left(\gamma + \frac{\gamma+1}{\tau}\right)\|\mathcal{F}(z_n^\delta) - y^\delta\|, \end{aligned}$$

and $\lim_{\beta\to 0} Q_n(\beta) \leq \left(\frac{1+\gamma}{\tau} + \gamma\right)\|\mathcal{F}(z_n^\delta) - y^\delta\|$.

As $\tau > \frac{1+\gamma}{\sigma-\gamma}$ we have existence of $\beta_n^\delta > 0$ for which (2.11) holds. □

2.4 Well defined-ness of method

Lemma 2.5. Let Assumption 2.1 be satisfied, and let

$$\theta := \sigma \left[(2-\sigma_0)\sigma - 2\left(\gamma + \frac{1+\gamma}{\tau}\right)\right] > 0,$$

then for any $0 \leq n < n_*$, we have

$$\|x_{n+1}^\delta - x^*\|^2 \leq \|z_n^\delta - x^*\|^2 - \frac{\sigma_n^\delta}{\beta_n^\delta}\theta\|r_n^\delta\|^2. \tag{2.15}$$

Proof. Note that $x_n^\delta, x_{n-1}^\delta \in \overline{\mathcal{B}}_{\rho/4}(x_0)$. Thus by the definition of z_n^δ and the fact that $\kappa_n^\delta \leq 1$, we obtain

$$\begin{aligned}
\|z_n^\delta - x_0\| &= \|z_n^\delta - x_n^\delta + x_n^\delta - x_0\| \\
&\leq \|z_n^\delta - x_n^\delta\| + \|x_n^\delta - x_0\| \\
&\leq \|x_n^\delta + \kappa_n^\delta(x_n^\delta - x_{n-1}^\delta) - x_n^\delta\| + \|x_n^\delta - x_0\| \\
&\leq \kappa_n^\delta\|x_n^\delta - x_{n-1}^\delta\| + \|x_n^\delta - x_0\| \\
&\leq \kappa_n^\delta\|(x_n^\delta - x^* + x^* - x_{n-1}^\delta)\| + \|x_n^\delta - x_0\| \\
&\leq \kappa_n^\delta\|x_n^\delta - x^*\| + \kappa_n^\delta\|x^* - x_{n-1}^\delta\| + \|x_n^\delta - x_0\| \\
&\leq \frac{1}{2}\kappa_n^\delta\rho + \frac{1}{2}\rho \leq \rho.
\end{aligned}$$

Thus we get $z_n^\delta \in \overline{\mathcal{B}}_\rho(x_0)$. Note that,

$$\|x_{n+1}^\delta - x^*\|^2 - \|z_n^\delta - x^*\|^2 = \langle z_n^\delta - \sigma_n^\delta u_n^\delta - x^*, z_n^\delta - \sigma_n^\delta u_n^\delta - x^*\rangle - \|z_n^\delta - x^*\|^2.$$

Using property of inner product, we get

$$\|x_{n+1}^\delta - x^*\|^2 - \|z_n^\delta - x^*\|^2 = 2\sigma_n^\delta\langle -u_n^\delta, z_n^\delta - x^*\rangle + (\sigma_n^\delta)^2\|u_n^\delta\|^2.$$

Using (2.7) and (2.12) we have

$$\|x_{n+1}^\delta - x^*\|^2 - \|z_n^\delta - x^*\|^2 = 2\sigma_n^\delta\langle -\mathcal{F}'(z_0^\delta)^*\nu_n^\delta, z_n^\delta - x^*\rangle + (\sigma_n^\delta)\frac{\sigma_0\langle\nu_n^\delta, r_n^\delta\rangle}{\|u_n^\delta\|^2}\|u_n^\delta\|^2$$

$$\|x_{n+1}^\delta - x^*\|^2 - \|z_n^\delta - x^*\|^2 \leq 2\sigma_n^\delta\langle\nu_n^\delta, -\mathcal{F}'(z_0^\delta)(z_n^\delta - x^*)\rangle + \sigma_0\sigma_n^\delta\langle\nu_n^\delta, r_n^\delta\rangle \tag{2.16}$$

It can be rewrite as,

$$-\mathcal{F}'(z_0^\delta)^*(z_n^\delta - x^*) = \mathcal{F}(z_n^\delta) - y - \mathcal{F}'(z_0^\delta)^*(z_n^\delta - x^*) - y^\delta + y - \mathcal{F}(z_n^\delta) + y^\delta.$$

Note that

$$\langle\nu_n^\delta, r_n^\delta\rangle = \|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-\frac{1}{2}}r_n^\delta\| \geq \sqrt{\beta_n^\delta}\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-1}r_n^\delta\|. \tag{2.17}$$

Using (2.7) in above inequality, we obtain

$$\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta I)^{-\frac{1}{2}}r_n^\delta\| \geq \frac{\sigma}{\sqrt{\beta_n^\delta}}\|r_n^\delta\|. \tag{2.18}$$

By equation (2.18) and (2.17) in (2.16) we can obtained as:

$$\begin{aligned}\|x_{n+1}^\delta - x^*\|^2 - \|z_n^\delta - x^*\|^2 &\leq 2\sigma_n^\delta\langle\nu_n^\delta, \mathcal{F}(z_n^\delta) - y - \mathcal{F}'(z_0^\delta)^*(z_n^\delta - x^*) - y^\delta + y \\ &\quad -\mathcal{F}(z_n^\delta) + y^\delta\rangle + \sigma_0\sigma_n^\delta\langle\nu_n^\delta, r_n^\delta\rangle \\ &\leq 2\sigma_n^\delta\langle\nu_n^\delta, \mathcal{F}(z_n^\delta) - y - \mathcal{F}'(z_0^\delta)(z_n^\delta - x^*)\rangle \\ &\quad +2\sigma_n^\delta\langle\nu_n^\delta, y - y^\delta\rangle - 2\sigma_n^\delta\langle\nu_n^\delta, r_n^\delta\rangle + \sigma_0\sigma_n^\delta\langle\nu_n^\delta, r_n^\delta\rangle.\end{aligned}$$

By Cauchy Schwarz inequality,

$$\begin{aligned}\|x_{n+1}^\delta - x^*\|^2 - \|z_n^\delta - x^*\|^2 &\leq 2\sigma_n^\delta\|\nu_n^\delta\|\|\mathcal{F}(z_n^\delta) - y - \mathcal{F}'(z_0^\delta)(z_n^\delta - x^*)\| \\ &\quad +2\sigma_n^\delta\|\nu_n^\delta\|\|y - y^\delta\| - 2\sigma_n^\delta\langle\nu_n^\delta, r_n^\delta\rangle + \sigma_0\sigma_n^\delta\langle\nu_n^\delta, r_n^\delta\rangle \\ &\leq 2\sigma_n^\delta\|\nu_n^\delta\|(\gamma\|\mathcal{F}(z_n^\delta) - y^\delta\|) + 2\sigma_n^\delta\|\nu_n^\delta\|\delta - \sigma_n^\delta(2-\sigma_0)\langle\nu_n^\delta, r_n^\delta\rangle \\ &\leq 2\sigma_n^\delta\|\nu_n^\delta\|(\gamma\|\mathcal{F}(z_n^\delta) - y^\delta + y^\delta - y\|) + 2\sigma_n^\delta\|\nu_n^\delta\|\delta \\ &\quad -\sigma_n^\delta(2-\sigma_0)\langle\nu_n^\delta, r_n^\delta\rangle \\ &\leq 2\sigma_n^\delta\|\nu_n^\delta\|(\gamma\|\mathcal{F}(z_n^\delta) - y^\delta\| + \gamma\|y^\delta - y\|) + 2\sigma_n^\delta\|\nu_n^\delta\|\delta \\ &\quad -\sigma_n^\delta(2-\sigma_0)\|(\mathcal{F}'(z_0^\delta)\|\mathcal{F}'(z_0^\delta)^* + \beta I)^{-\frac{1}{2}}r_n^\delta\|^2 \\ &\leq 2\sigma_n^\delta\|\nu_n^\delta\|(\gamma\|r_n^\delta\| + (1+\gamma)\delta) \\ &\quad -\sigma_n^\delta(2-\sigma_0)\|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I)^{-\frac{1}{2}}r_n^\delta\|^2. \\ &\leq 2\sigma_n^\delta\frac{\sigma}{\beta_n^\delta}\left(\gamma + \frac{1+\gamma}{\tau}\right)\|r_n^\delta\|^2 - \sigma_n^\delta\frac{(2-\sigma_0)\sigma^2}{\beta_n^\delta}\|r_n^\delta\|^2 \\ &\leq \sigma_n^\delta\frac{\sigma}{\beta_n^\delta}\|r_n^\delta\|^2\left\{2\left(\gamma + \frac{1+\gamma}{\tau}\right) - (2-\sigma_0)\sigma\right\} \\ &\leq -\frac{\sigma_n^\delta}{\beta_n^\delta}\theta\|r_n^\delta\|^2.\end{aligned}$$

Thus, we obtain,

$$\|x_{n+1}^\delta - x^*\|^2 \leq \|z_n^\delta - x^*\|^2 - \frac{\sigma_n^\delta}{\beta_n^\delta}\theta\|r_n^\delta\|^2.$$

Hence proof is completed. □

Proposition 2.6. Suppose Assumption 2.1 and (2.9) hold. Let

$$\xi : \min\left\{\frac{\sigma_0}{\hat{\mathcal{L}}^2}, \frac{M}{J_2}\right\} \times \theta > 0.$$

and

$$\Delta_n^\delta := \|x_n^\delta - x^*\|^2 - \|x_{n-1}^\delta - x^*\|^2$$

Then for any $0 \leq n < n_*$, the following inequality hold.

$$\Delta_{n+1}^\delta = \kappa_n^\delta\Delta_n^\delta + \kappa_n^\delta(\kappa_n^\delta + 1)\|x_n^\delta - x_{n-1}^\delta\|^2 - \xi\|r_n^\delta\|^2. \tag{2.19}$$

Proof. Note that

$$\frac{\langle\nu_n^\delta, r_n^\delta\rangle}{\|u_n^\delta\|^2} = \frac{\langle\nu_n^\delta, r_n^\delta\rangle}{\langle u_n^\delta, u_n^\delta\rangle},$$

Taking $u_n^\delta = \mathcal{F}'(z_0^\delta)^*\nu_n^\delta$ in the above equation, we get

$$\frac{\langle\nu_n^\delta, r_n^\delta\rangle}{\|u_n^\delta\|^2} = \frac{\langle\nu_n^\delta, r_n^\delta\rangle}{\langle\mathcal{F}'(z_0^\delta)^*\nu_n^\delta, \mathcal{F}'(z_0^\delta)^*\nu_n^\delta\rangle}$$

which imply

$$\frac{\langle \nu_n^\delta, r_n^\delta \rangle}{\|u_n^\delta\|^2} = \frac{\langle \nu_n^\delta, r_n^\delta \rangle}{\langle \mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* \nu_n^\delta, \nu_n^\delta \rangle}.$$

By Cauchy Schwarz inequality, we can write

$$\frac{\langle \nu_n^\delta, r_n^\delta \rangle}{\|u_n^\delta\|^2} \geq \frac{\|\nu_n^\delta\| \|r_n^\delta\|}{\tilde{\mathcal{L}}^2 \|\nu_n^\delta\|^2}. \tag{2.20}$$

From (2.11) we have

$$\beta_n^\delta \|(\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I)^{-1}(\mathcal{F}(z_n^\delta) - y^\delta)\| = \sigma \|\mathcal{F}(z_n^\delta) - y^\delta)\|,$$

Thus we have

$$\frac{\beta_n^\delta \|\nu_n^\delta\|}{\sigma} = \|r_n^\delta\|, \tag{2.21}$$

where $\nu_n^\delta = (\mathcal{F}'(z_0^\delta)\mathcal{F}'(z_0^\delta)^* + \beta_n^\delta I)^{-1}(\mathcal{F}(z_n^\delta) - y^\delta)$ and $r_n^\delta = \mathcal{F}(z_n^\delta) - y^\delta)$. Thus from (2.20) and (2.21), we get

$$\frac{\langle \nu_n^\delta, r_n^\delta \rangle}{\|u_n^\delta\|^2} \geq \frac{\|\nu_n^\delta\| \beta_n^\delta \|\nu_n^\delta\|}{\sigma \tilde{\mathcal{L}}^2 \|\nu_n^\delta\|^2},$$

which implies

$$\frac{\langle \nu_n^\delta, r_n^\delta \rangle}{\|u_n^\delta\|^2} \geq \frac{\beta_n^\delta}{\sigma \tilde{\mathcal{L}}^2}.$$

$$\sigma_n^\delta := \min\left\{ \frac{\sigma_0 \langle \nu_n^\delta, r_n^\delta \rangle}{\|u_n^\delta\|^2}, M \right) \geq \min\left\{ \frac{\sigma_0 \beta_n^\delta}{\tilde{\mathcal{L}}^2}, M \right\}$$

$$\frac{\sigma_n^\delta}{\beta_n^\delta} \geq \min\left\{ \frac{\sigma_0}{\tilde{\mathcal{L}}^2}, \frac{M}{\beta_n^\delta} \right\} \geq \min\left\{ \frac{\sigma_0}{\tilde{\mathcal{L}}^2}, \frac{M}{J_2} \right\},$$

where $\beta_n^\delta \geq J_2$ and multiplying on both side by θ above equation then we get;

$$\frac{\sigma_n^\delta}{\beta_n^\delta}\theta \geq \min\left\{ \frac{\sigma_0}{\tilde{\mathcal{L}}^2}, \frac{M}{J_2} \right\} \theta = \xi \geq 0. \tag{2.22}$$

Also we know that,

$$\|x_{n+1}^\delta - x^*\|^2 - \|z_n^\delta - x^*\|^2 \leq -\frac{\sigma_n^\delta}{\beta_n^\delta}\theta \|r_n^\delta\|^2,$$

by above equation (2.22) we have

$$\|x_{n+1}^\delta - x^*\|^2 \leq \|z_n^\delta - x^*\|^2 - \xi \|r_n^\delta\|^2.$$

Now given that

$$\Delta_n^\delta = \|x_n^\delta - x^*\|^2 - \|x_{n-1}^\delta - x^*\|^2$$

$$\begin{aligned}
\Delta_{n+1}^\delta &= \|x_{n+1}^\delta - x^*\|^2 - \|x_n^\delta - x^*\|^2 \\
&\leq \|z_n^\delta - x^*\|^2 - \|x_n^\delta - x^*\|^2 - \xi \|r_n^\delta\|^2. \\
&\leq \|x_n^\delta + \kappa_n^\delta (x_n^\delta - x_{n-1}^\delta) - x^*\|^2 - \|x_n^\delta - x^*\|^2 - \xi \|r_n^\delta\|^2. \\
&\leq \|x_n^\delta - x^*\|^2 + (\kappa_n^\delta)^2 \|x_n^\delta - x_{n-1}^\delta\|^2 + 2\|\kappa_n^\delta (x_n^\delta - x_{n-1}^\delta)(x_n^\delta - x^*)\| - \|x_n^\delta - x^*\|^2 - \xi \|r_n^\delta\|^2. \\
&\leq (\kappa_n^\delta)^2 \|x_n^\delta - x_{n-1}^\delta\|^2 + 2\|\kappa_n^\delta (x_n^\delta - x_{n-1}^\delta)(x_n^\delta - x^*)\| - \xi \|r_n^\delta\|^2. \\
&\leq (\kappa_n^\delta)^2 \|x_n^\delta - x_{n-1}^\delta\|^2 + 2\kappa_n^\delta \langle x_{n-1}^\delta - x_n^\delta, x_n^\delta - x^* \rangle - \xi \|r_n^\delta\|^2. \\
&\leq (\kappa_n^\delta)^2 \|x_n^\delta - x_{n-1}^\delta\|^2 - \kappa_n^\delta (\|x_{n-1}^\delta - x^*\|^2 - \|x_n^\delta - x^*\|^2 - \|x_n^\delta - x_{n-1}^\delta\|^2) - \xi \|r_n^\delta\|^2. \\
&\leq (\kappa_n^\delta)^2 \|x_n^\delta - x_{n-1}^\delta\|^2 - \kappa_n^\delta \|x_{n-1}^\delta - x^*\|^2 + \kappa_n^\delta \|x_n^\delta - x^*\|^2 + \kappa_n^\delta \|x_n^\delta - x_{n-1}^\delta\|^2 - \xi \|r_n^\delta\|^2. \\
&\leq \kappa_n^\delta \Delta_n^\delta + \kappa_n^\delta (1 + \kappa_n^\delta) \|x_n^\delta - x_{n-1}^\delta\|^2 - \xi \|r_n^\delta\|^2.
\end{aligned}$$

Thus proof is completed. □

Proposition 2.7. Let $\vartheta > 1$ and let κ_n^δ be chosen such that:

$$\kappa_n^\delta(\kappa_n^\delta + 1)\|x_n^\delta - x_{n-1}^\delta\|^2 - \frac{\xi\|r_n^\delta\|^2}{\vartheta} \leq 0, \tag{2.23}$$

Then following hold.

(a) $\|x_{n+1}^\delta - x^*\|^2 \leq \|x_n^\delta - x^*\|^2$ and $x_n^\delta \in \overline{\mathcal{B}}_\rho(x_0), \forall 0 \leq n < n_*$;

(b) $\sum_{n=0}^{n_*} \|\mathcal{F}(z_n^\delta) - y^\delta\|^2 < \infty$;

(c) The integer n_*, given in (1.5), is finite.

Proof. We proceed with the proof by induction. Given that $x_{-1}^\delta = x_0^\delta$ it follows from equation (2.19), together with the condition (2.23), that

$$\Delta_{n+1}^\delta \leq \kappa_n^\delta \Delta_n^\delta + \kappa_n^\delta(1 + \kappa_n^\delta)\|x_n^\delta - x_{n-1}^\delta\|^2 - \xi\|r_n^\delta\|^2,$$

For $n = 0$, we have

$$\Delta_1^\delta \leq \kappa_0^\delta \Delta_0^\delta + \kappa_0^\delta(1 + \kappa_0^\delta)\|x_0^\delta - x_{0-1}^\delta\|^2 - \xi\|r_0^\delta\|^2,$$
$$\Delta_1^\delta \leq \kappa_0^\delta \Delta_0^\delta. \tag{2.24}$$

We know that

$$\Delta_n^\delta := \|x_n^\delta - x^*\|^2 - \|x_{n-1}^\delta - x^*\|^2.$$

For $n = 0$, we note

$$\Delta_0^\delta := \|x_0^\delta - x^*\|^2 - \|x_{-1}^\delta - x^*\|^2 = 0 \tag{2.25}$$

Using (2.24) and (2.25) we have

$$\Delta_1^\delta \leq 0.$$

We assume the result hold for the case $n = m$, we will proof it for $n = m + 1$, which will show that there holds for every $0 \leq n \leq n_*$.
Taking $n = m$ in equation (2.19), we have:

$$\Delta_{m+1}^\delta \leq \kappa_m^\delta \Delta_m^\delta + \kappa_m^\delta(1 + \kappa_m^\delta)\|x_m^\delta - x_{m-1}^\delta\|^2 - \xi\|r_m^\delta\|^2. \tag{2.26}$$

Puting $n = m$ in equation (2.23), we get

$$\kappa_m^\delta(\kappa_m^\delta + 1)\|x_m^\delta - x_{m-1}^\delta\|^2 - \frac{\xi\|r_m^\delta\|^2}{\vartheta} \leq 0.$$

Equivalenty, we can write

$$\kappa_m^\delta(\kappa_m^\delta + 1)\|x_m^\delta - x_{m-1}^\delta\|^2 \leq \frac{\xi\|r_m^\delta\|^2}{\vartheta}. \tag{2.27}$$

From (2.26) and (2.27), we get

$$\Delta_{m+1}^\delta \leq \kappa_m^\delta \Delta_m^\delta + \frac{\xi\|r_m^\delta\|^2}{\vartheta} - \xi\|r_m^\delta\|^2$$

$$\Delta_{m+1}^\delta \leq \kappa_m^\delta \Delta_m^\delta + \frac{1 - \vartheta}{\vartheta}\xi\|r_m^\delta\|^2$$

$$\Delta_{m+1}^\delta \leq 0.$$

So for every n, we have

$$\Delta_{n+1}^{\delta} \leq \kappa_n^{\delta}\Delta_n^{\delta} \leq \cdots \leq \kappa_n^{\delta}\kappa_{n-1}^{\delta}\cdots \leq \kappa_1^{\delta}\Delta_1^{\delta} \leq \kappa_0^{\delta}\Delta_0^{\delta}.$$

Hence,

$$\Delta_{n+1}^{\delta} \leq \kappa_0^{\delta}\Delta_0^{\delta};$$

which implies,

$$\Delta_{n+1}^{\delta} \leq 0, \quad \forall \quad 0 \leq n < n^*, \tag{2.28}$$

Moreover,

$$\|x_{n+1}^{\delta} - x^*\|^2 \leq \|x_n^{\delta} - x^*\|^2 \leq \cdots \|x_0^{\delta} - x^*\|^2 \leq (\rho/4)^2$$

This directly implies that

$$\|x_{n+1}^{\delta} - x^*\|^2 \leq \|x_n^{\delta} - x^*\|^2 \leq \cdots \|x_0^{\delta} - x^*\|^2 \leq (\rho/4)^2 \leq \rho$$

i.e, $x_n^{\delta} \in \overline{\mathcal{B}}_{\rho}(x_0)$, hence, (a) holds.
From to (2.19), (2.23) and (2.28), we have

$$\Delta_{n+1}^{\delta} \leq \frac{1}{\vartheta}\xi\|r_n^{\delta}\|^2 - \xi\|r_n^{\delta}\|^2.$$

Therefore

$$\overline{\vartheta}\xi\|r_n^{\delta}\|^2 \leq \|x_n^{\delta} - x^*\|^2 - \|x_{n+1}^{\delta} - x^*\|^2$$

with $\bar{\vartheta} := \frac{\vartheta-1}{\vartheta} > 0$. As a result,

$$\sum_{n=0}^{n_*-1} \|\mathcal{F}(z_n^{\delta}) - y^{\delta}\|^2 \leq \left(\frac{1}{\overline{\vartheta}\xi}\right) \sum_{n=0}^{n_*-1} \|x_n^{\delta} - x^*\|^2 - \|x_{n+1}^{\delta} - x^*\|^2$$

If n_* is infinite, then $\|\mathcal{F}(z_n^{\delta}) - y^{\delta}\| \geq \tau\delta$ for all $n \in \mathbb{N}$. As $x^* \in \mathcal{B}_{\rho}(x_0)$, we have

$$\sum_{n=0}^{\infty} \|\mathcal{F}(z_n^{\delta}) - y^{\delta}\|^2 \leq \frac{1}{\overline{\vartheta}\xi}\|x_0^{\delta} - x_*\|^2 \leq \frac{1}{\overline{\vartheta}\xi}\rho^2 < \infty,$$

which implies

$$\lim_{n\to\infty} \|\mathcal{F}(z_n^{\delta}) - y^{\delta}\| = 0. \tag{2.29}$$

This contradicts $\|\mathcal{F}(z_n^{\delta}) - y^{\delta}\| \geq \tau\delta > 0$. Hence,

$$\sum_{n=0}^{n_*} \|\mathcal{F}(z_n^{\delta}) - y^{\delta}\|^2 \leq \infty,$$

which implies n_* is finite. This complete the proof. □

2.5 Choice of combination parameter κ_n^{δ}

We choose κ_n^{δ} as follows:

$$\kappa_n^{\delta} = \min\left(\sqrt{\frac{\mu(\tau\delta)^2}{\mu\|x_n^{\delta} - x_{n-1}^{\delta}\|^2} + \frac{1}{4}} - \frac{1}{2}, \frac{n-1}{n+\omega-1}\right). \tag{2.30}$$

2.6 Convergence analysis in noise free case

In this section, we will following notation for similarity:

$$r_n := \mathcal{F}(z_n) - y,$$

$$\nu_n := (\mathcal{F}'(z_0)\mathcal{F}'(z_0)^* + \beta_n I)^{-1} r_n, \tag{2.31}$$

$$u_n := \mathcal{F}'(z_0)^* \nu_n, \tag{2.32}$$

Let $x_0 = z_0 \in X$. Then we define

$$z_n = x_n + \kappa_n(x_n - x_{n-1}) \tag{2.33}$$

$$x_{n+1} = z_n - \sigma_n \mathcal{F}'(z_0)^* \left(\mathcal{F}'(z_0)\mathcal{F}'(z_0)^* + \beta_n I\right)^{-1} \left(\mathcal{F}(z_n) - y\right), \tag{2.34}$$

where the stepsize σ_n chosen such that

$$\sigma_n := \min\left\{\frac{\sigma_0 \langle \nu_n, r_n\rangle}{\|u_n\|^2}, M\right\}. \tag{2.35}$$

we choose the regularization parameter $\beta_n > 0$ which satisfy

$$\beta_n \|(\mathcal{F}'(z_0)\mathcal{F}'(z_0)^* + \beta_n I)^{-1}(\mathcal{F}(z_n) - y)\| = \sigma \|\mathcal{F}(z_n) - y)\|, \tag{2.36}$$

while $0 < \sigma < 1$. In noise free case we assume.

$$\sum_{n=0}^{\infty} \kappa_n \|x_n - x_{n-1}\| < \infty. \tag{2.37}$$

Lemma 2.8. Let Assumption 2.1 be hold and let κ_n be chosen as

$$\kappa_n(\kappa_n + 1)\|x_n - x_{n-1}\|^2 - \frac{\xi \|r_n\|^2}{\vartheta} \leq 0, \tag{2.38}$$

(i) $\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2$ and $x_n \in \overline{\mathcal{B}}_\rho(x_0)$ for all $n \geq 0$;

(ii) $\sum_{n=0}^{\infty} \|\mathcal{F}(z_n) - y\|^2 < \infty$.

Proof. We will use the induction hypothesis to prove the result. Given $x_{-1} = x_0$, it follows from equation (2.19), together with the condition (2.38), that Let

$$\Delta_{n+1} \leq \kappa_n \Delta_n + \kappa_n(1 + \kappa_n)\|x_n - x_{n-1}\|^2 - \xi \|r_n\|^2.$$

For $n = 0$, we have

$$\Delta_1 \leq \kappa_0 \Delta_0 + \kappa_0(1 + \kappa_0)\|x_0^\delta - x_{0-1}\|^2 - \xi \|r_0^\delta\|^2.$$

$$\Delta_1 \leq \kappa_0 \Delta_0 \tag{2.39}$$

We know that

$$\Delta_n := \|x_n - x^*\|^2 - \|x_{n-1} - x^*\|^2$$

For $n = 0$ we note

$$\Delta_0 := \|x - x^*\|^2 - \|x_{-1} - x^*\|^2 = 0. \tag{2.40}$$

Using (2.39) and (2.40) we have

$$\Delta_1 \leq 0.$$

We assume the result hold for the case $n = m$, we will proof it for $n = m + 1$, which will show the there holds for every n.
Taking $n = m$ in equation (2.39), we have

$$\kappa_m(\kappa_m + 1)\|x_m - x_{m-1}\|^2 - \frac{\xi\|r_m\|^2}{\vartheta} \leq 0, \tag{2.41}$$

Putting $n = m$ in the equation (2.19), we get,

$$\Delta_{m+1} \leq \kappa_m\Delta_m + \kappa_m(1 + \kappa_m)\|x_m - x_{m-1}\|^2 - \xi\|r_m\|^2. \tag{2.42}$$

From equation (2.41) and (2.42) we get

$$\Delta_{m+1} \leq \kappa_m\Delta_m + \frac{\xi\|r_m\|^2}{\vartheta} - \xi\|r_m\|^2.$$

which implies

$$\Delta_{m+1} \leq \kappa_m\Delta_m + \frac{1-\vartheta}{\vartheta}\xi\|r_m\|^2$$

Hence,

$$\Delta_{m+1} \leq 0.$$

So for every n, we have

$$\Delta_{n+1} \leq \kappa_n\Delta_n \leq \cdots \leq \kappa_n\kappa_{n-1}\cdots \leq \kappa_1\Delta_1 \leq \kappa_0\Delta_0.$$

Hence,

$$\Delta_{n+1} \leq \kappa_0\Delta_0,$$

which shows

$$\Delta_{n+1} \leq 0,$$

Moreover,

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 \leq \cdots \|x_0 - x^*\|^2 \leq (\rho/4)^2.$$

This directly implies that

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 \leq \cdots \|x_0 - x^*\|^2 \leq (\rho/4)^2 \leq \rho$$

i.e. $x_n \in \overline{\mathcal{B}}_\rho(x_0)$. Hence, (i) hold.
Now,from (2.19), (2.38) and (2.28), there holds:

$$\Delta_{n+1} \leq \frac{1}{\vartheta}\xi\|r_n\|^2 - \xi\|r_n\|^2.$$

Therefore

$$\overline{\vartheta}\xi\|r_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

with $\bar{\vartheta} := \frac{\vartheta-1}{\vartheta} > 0$. As a result,

$$\sum_{n=0}^{\infty}\|\mathcal{F}(z_n) - y^\delta\|^2 \leq \frac{1}{\overline{\vartheta}\xi}\sum_{n=0}^{\infty}\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

As $x^* \in \mathcal{B}_\rho(x_0)$, we have

$$\sum_{n=0}^{\infty}\|\mathcal{F}(z_n^\delta) - y^\delta\|^2 \leq \frac{1}{\overline{\vartheta}\xi}\|x_0^\delta - x_*\|^2 \leq \frac{1}{\overline{\vartheta}\xi}\rho^2 < \infty.$$

This completes the proof.

□

Lemma 2.9. Let $\{x_n\}$ and $\{z_n\}$ are as given in (2.33) and (2.34). Then

$$x_i - x_l = \sum_{j=l}^{i-1} \kappa_j (x_j - x_{j-1}) - \sum_{j=l}^{i-1} \sigma_j u_j$$

hold for all $i,\ l \in \mathbb{N}$.

Proof. From (2.32) and (2.34), we have

$$\begin{aligned}
x_i - x_l &= z_{i-1} - \sigma_{i-1}u_{i-1} - x_l \\
&= x_{i-1} + \kappa_{i-1}(x_{i-1} - x_{i-2}) - \sigma_{i-1}u_{i-1} - x_l \\
&= z_{i-2} - \sigma_{i-2}u_{i-2} + \kappa_{i-1}(x_{i-1} - x_{i-2}) - \sigma_{i-1}u_{i-1} - x_l \\
&= x_{i-2} + \kappa_{i-2}(x_{i-2} - x_{i-3}) - \sigma_{i-2}u_{i-2} + \kappa_{i-1}(x_{i-1} - x_{i-2}) - \sigma_{i-1}u_{i-1} - x_l \\
&= x_{i-2} + \sum_{j=i-2}^{i-1} \kappa_j (x_j - x_{j-1}) - \sum_{j=i-2}^{i-1} \sigma_j u_j - x_l \\
&= x_l + \sum_{j=l}^{i-1} \kappa_j (x_j - x_{j-1}) - \sum_{j=l}^{i-1} \sigma_j u_j - x_l
\end{aligned}$$

Hence, we prove that

$$x_i - x_l = \sum_{j=l}^{i-1} \kappa_j (x_j - x_{j-1}) - \sum_{j=l}^{i-1} \sigma_j u_j.$$

□

Lemma 2.10. Let $\{x_n\}$ and $\{z_n\}$ be sequences as given in (2.33) and (2.34), respectively. Then $\|x_n - x^*\|$ converges as $n \to \infty$ and limits of sequence $\|z_n - x^*\|$ and $\|x_n - x^*\|$ will be same.

Proof. From Lemma 2.8, we have $\{\|x_n - x^*\|\}$ is a bounded and monotonic sequence, which implies $\{\|x_n - x^*\|\}$ is a convergent sequence. Definition of z_n in (2.33) implies

$$\begin{aligned}
\|z_n - x^*\| &= \|x_n + \kappa_n(x_n - x_{n-1}) - x^*\| \\
&\leq \|x_n - x^*\| + \kappa_n \|x_n - x_{n-1}\|
\end{aligned}$$

Taking limit $n \to \infty$, both side of above inequality, we get

$$\lim_{n\to\infty} \|z_n - x^*\| \leq \lim_{n\to\infty} \left(\|x_n - x^*\| + \kappa_n \|x_n - x_{n-1}\|\right).$$

Thus, we have

$$\lim_{n\to\infty} \|z_n - x_*\| \leq \lim_{n\to\infty} \|x_n - x^*\|.$$

Similarlly, we can show $\lim_{n\to\infty} \|x_n - x_*\| \leq \lim_{n\to\infty} \|z_n - x^*\|$.
This proof that $\|z_n - x^*\|$ and $\|x_n - x^*\|$ converges to same limit. □

Lemma 2.11. Suppose Assumption 2.1 satisfy. If $\mathcal{F}(z_n) = y$ for some $n \geq 1$, then $\mathcal{F}(x_0) = y$.

Proof. We show $\mathcal{F}(z_{n-1}) = y$ if $\mathcal{F}(z_n) = y$ for some $n \geq 1$. We will use contradiction technique to prove it. Let $\mathcal{F}(z_{n-1}) \neq y$. By definition of z_n and equation (2.32), we have

$$\|z_n - z_{n-1}\| = \| - x_n - \sigma_{n-1}u_{n-1} + x_n + \kappa_n(x_n - x_{n-1})\|.$$

Therefore,

$$\|z_n - z_{n-1}\| = \| - \sigma_{n-1}u_{n-1} + \kappa_n(x_n - x_{n-1})\|.$$

By traingle inequality, we have

$$\begin{aligned} \|z_n - z_{n-1}\| &\leq \sigma_{n-1}\|u_{n-1}\| + \kappa_n\|(x_n - x_{n-1})\| \\ &\leq \sigma_{n-1}\|\mathcal{F}'(z_0)^*\nu_{n-1}\| + \kappa_n\|(x_n - x_{n-1})\| \\ &\leq \sigma_{n-1}\hat{\mathcal{L}}\|\nu_{n-1}\| + \kappa_n\|(x_n - x_{n-1})\|, \end{aligned}$$

$$\leq M\hat{\mathcal{L}}\|\nu_{n-1}\| + \kappa_n\|(x_n - x_{n-1})\|, \tag{2.43}$$

where $\hat{\mathcal{L}}$ is a constant such that $\|\mathcal{F}'(z_0)^*\| \leq \hat{\mathcal{L}}$ and $u_n = \mathcal{F}'(z_0)^*\nu_n$.
As $\mathcal{F}(z_n) = y$ implies $r_n = 0$ which gives $\|r_n\| = 0$. So, we have $\|x_n - x_{n-1}\| = 0$, using this in (2.43), we have

$$\|z_n - z_{n-1}\| \leq \frac{\sigma M\hat{\mathcal{L}}}{\beta_{n-1}}\|r_{n-1}\| \leq \frac{\sigma M\hat{\mathcal{L}}}{J_1}\|r_{n-1}\| \tag{2.44}$$

Again from Proposition 2.2, we have

$$\|z_n - z_{n-1}\| \geq \frac{1}{\hat{\mathcal{L}}}\|\mathcal{F}'(z_0)(z_n - z_{n-1})\| \geq \frac{1-\gamma}{\hat{\mathcal{L}}}\|\mathcal{F}(z_n) - \mathcal{F}(z_{n-1})\| \geq \frac{1-\gamma}{\hat{\mathcal{L}}}\|\mathcal{F}(z_{n-1}) - y\|. \tag{2.45}$$

Thus from (2.44) and (2.45), we get

$$\frac{1-\gamma}{\hat{\mathcal{L}}}\|r_{n-1}\| \leq \|z_n - z_{n-1}\| \leq \frac{\sigma M\hat{\mathcal{L}}}{J_1}\|r_{n-1}\|.$$

Hence,

$$\frac{1-\gamma}{\hat{\mathcal{L}}}\|r_{n-1}\| \leq \frac{\sigma M\hat{\mathcal{L}}}{J_1}\|r_{n-1}\|,$$

which implies

$$\frac{1-\gamma}{\hat{\mathcal{L}}} \leq \frac{\sigma M\hat{\mathcal{L}}}{J_1}.$$

Thus

$$M \geq \frac{J_2(1-\gamma)}{\hat{\mathcal{L}}^2\sigma} = \frac{\mathcal{L}(1-\gamma)}{\hat{\mathcal{L}}},$$

which is contradiction to (2.9), Hence $\mathcal{F}(x_{n-1}) = y$ which gives $\mathcal{F}(x_0) = y$. □

In the following result, we show convergence of method in case of noise free data.

Theorem 2.12. Let Assumption 2.1 be hold . Then the sequences $\{z_n\}$ and $\{x_n\}$ converge strongly to a solution of (1.1).

Proof. Let $v_n := x^* - z_n$. From Lemma 2.10, we note that $\|v_n\|$ is convergence. Now we show sequence $\{z_n\}$ is Cauchy for this we show $\{v_n\}$ is a Cauchy sequence. We select some integers $l = l(n,i) \in \{n, n+1, ..., i\}$ for $i \geq n$, satisfying

$$\|\mathcal{F}(z_l) - y\| \leq \|\mathcal{F}(z_j) - y\|, \forall n \leq j \leq i. \tag{2.46}$$

From traingle inequlity, we have

$$\|v_i - v_n\| \leq \|v_i - v_l\| + \|v_l - v_n\|. \tag{2.47}$$

Using inner product definition, we get

$$\|v_i - v_l\|^2 = \|v_i\|^2 - \|v_l\|^2 + 2\langle v_l - v_i, v_l\rangle, \quad \|v_l - v_n\|^2 = \|v_n\|^2 - \|v_l\|^2 + 2\langle v_l - v_n, v_l\rangle. \tag{2.48}$$

We shall show

$$|\langle v_l - v_i, v_l\rangle| \to 0, \quad |\langle v_l - v_n, v_l\rangle| \to 0, \tag{2.49}$$

as $n, j \to \infty$
From the first term of equation (2.49), it follow that

$$\begin{aligned}
|\langle v_l - v_i, v_l\rangle| &= |\langle x^* - z_l - x^* + z_i, v_l\rangle| \\
&= |\langle z_i - z_l, v_l\rangle| \\
&= |\langle x_i + \kappa_i(x_i - x_{i-1}) - x_l - \kappa_l(x_l - x_{l-1}), v_l\rangle| \\
&\leq |\langle x_i - x_l, v_l\rangle| + \kappa_i |\langle x_i - x_{i-1}, v_l\rangle| + \kappa_l |\langle x_l - x_{l-1}, v_l\rangle| .
\end{aligned}$$

By Cauchy-Schwarz inequality,

$$|\langle v_l - v_i, v_l\rangle| \leq |\langle x_i - x_l, v_l\rangle| + \kappa_i \|x_j - x_{i-1}\| \|v_l\| + \kappa_l \|x_l - x_{l-1}\| \|v_l\|.$$

Using boundedness of $\|v_n\|$ and Lemma 2.8, we get

$$(\kappa_i \|x_i - x_{i-1}\| \|v_l\| + \kappa_l \|x_l - x_{l-1}\| \|v_l\|) \to 0, \quad \text{as} \quad n \to \infty \tag{2.50}$$

Now, we show that $|\langle x_i - x_l, v_l\rangle| \to 0 \quad \text{as} \quad n \to \infty$. Using Lemma 2.9, it can be seen that

$$\begin{aligned}
|\langle x_j - x_l, v_l\rangle| &= \left| \left\langle \sum_{j=l}^{i-1} \kappa_j (x_j - x_{j-1}) - \sum_{j=l}^{i-1} \sigma_j u_j, v_l \right\rangle \right| \\
&\leq \sum_{j=l}^{i-1} \kappa_j |\langle x_j - x_{j-1}, v_l\rangle| + \sum_{j=l}^{i-1} \sigma_j |\langle u_j, v_l\rangle|
\end{aligned}$$

Moreover,

$$\sum_{j=l}^{i-1} \kappa_j |\langle x_j - x_{j-1}, v_l\rangle| \leq \sum_{j=l}^{i-1} \kappa_j \|x_j - x_{j-1}\| \|v_l\| \leq \sum_{j=l}^{\infty} \kappa_j \|x_j - x_{j-1}\| \|v_l\| \to 0 \tag{2.51}$$

as $n \to \infty$. Now, let as consider

$$\sum_{j=l}^{i-1} \sigma_j |\langle u_j, v_l\rangle| = \sum_{j=l}^{i-1} \sigma_j |\langle \mathcal{F}'(z_0)\nu_j, v_l\rangle|$$

where $u_j = \mathcal{F}'(z_0)^* \nu_j$, using the fact that $\sigma_j < M$, we have

$$\sum_{j=l}^{i-1} \sigma_j |\langle u_j, v_l\rangle| < \sum_{j=l}^{i-1} M |\langle \nu_j, \mathcal{F}'(z_0)(x_* - z_l)\rangle| .$$

By Cauchy-Schwarz inequality,

$$\sum_{j=l}^{i-1} \sigma_j |\langle u_j, v_l\rangle| \leq M \sum_{j=l}^{i-1} \|\nu_j\| \|\mathcal{F}'(z_0)(x_* - z_l)\|.$$

From the Proposition (2.1) and equation (2.28), we have:

$$\sum_{j=l}^{i-1}\sigma_j\left|\langle u_j,v_l\rangle\right| \le M\sum_{j=l}^{i-1}\frac{\sigma}{\beta_j}\|\mathcal{F}(z_j)-y\|\left\{2(1+\gamma)\|\mathcal{F}(z_j)-y\|+(1+\gamma)\|\mathcal{F}(z_l)-y\|\right\}$$
$$\le M(1+\gamma)\frac{\sigma}{\beta_j}\sum_{j=l}^{i-1}\left\{2\|\mathcal{F}(z_i)-y\|^2+\|\mathcal{F}(z_j)-y\|\|\mathcal{F}(z_l)-y\|\right\}.$$

From equation (2.13) and (2.46), we have:

$$\sum_{j=l}^{i-1}\sigma_j\left|\langle u_j,v_l\rangle\right| \le \frac{3M(1+\gamma)}{\tilde{\mathcal{L}}}\sum_{j=l}^{i-1}\|\mathcal{F}(z_j)-y\|^2 \le \frac{3M(1+\gamma)}{\tilde{\mathcal{L}}}\sum_{j=l}^{\infty}\|\mathcal{F}(z_j)-y\|^2. \tag{2.52}$$

From Lemma 2.8, we have $\lim_{n\to\infty}\|\mathcal{F}(z_j)-y\|=0$, so

$$\sum_{j=l}^{i-1}\sigma_j\left|\langle u_j,v_l\rangle\right| \to 0 \quad \text{as} \quad n\to\infty. \tag{2.53}$$

Using equations (2.51) and (2.53) in (2.48), we get $|\langle x_i-x_l,v_l\rangle|\to 0$ as $n\to\infty$.
Similarly we can show $|\langle x_l-x_n,v_l\rangle|\to 0$ as $n\to\infty$. Thus, $\{v_n\}$ is Cauchy sequance. This shows sequence $\{z_n\}$ is Cauchy sequence and hence converges . The condition $\|\mathcal{F}(z_n)-y\|\to 0$ implies that the sequence $\{z_n\}$ converges to a solution of (1.1). Again by Lemma 2.10, we can conclude that $\{x_n\}$ and $\{z_n\}$ converge to same limit. □

2.7 Asymptotically stability and Regularization property

Let $\{\delta_k\}$ be a sequence of positive real numbers converges to zero. The associated stopping index and iteration points are represented by

$$\hat{n}^k := n(\delta_k, y^{\delta_k}), \quad x^k := x^{\delta_k}_{\hat{n}_k}, \quad z^k := z^{\delta_k}_{\hat{n}_k},$$

respectively. Let combination parameters be denoted by $\kappa^k := \kappa^{\delta_k}_{\hat{n}_k}, and \sigma^k := \sigma^{\delta_k}_{\hat{n}_k}$.

We state a result, which gives convergence in a weak sense and its proof is done using similar techniques as in [3]

Proposition 2.13. Suppose Assumption 2.1 and equation (2.9) are satisfied. Then any subsequences of the sequences $\{x^k\}$ and $\{z^k\}$ have a subsequence which converges weakly to the solution of (1.1). Furthermore, if $x^\dagger \in \bar{\mathcal{B}}_\rho(x_0)$ is the unique solution of (1.1), then both sequences $\{x^k\}$ and $\{z^k\}$ are convergent weakly to $x^\dagger$.

Proof. We can assume without loss of generality that δ_k itself is a subsequence. As sequence $\{x^k\}$ is bounded in X, there exist a subsequence $\{x^k\}$ which is defined by same notation and an element $\bar{x}\in X$ such that.

$$x^k \rightharpoonup \bar{x} \quad weakly, \quad z^k \to \bar{z} \quad weakly \quad as \quad k\to\infty.$$

From Assumption 2.1 we note that $\mathcal{F}$ is completely continuous. So,

$$\mathcal{F}(x^k)\to\mathcal{F}(\bar{x}), \quad and \quad \mathcal{F}(z^k)\to\mathcal{F}(\bar{z}) \quad as \quad k\to\infty.$$

Thus, $y_k^\delta - \mathcal{F}(z^k) \to y^\dagger - \mathcal{F}(\bar{z})$ in Y. By Morozov's discrepancy principle $\|y_k^\delta - \mathcal{F}(z^k)\| \leq \tau\delta_k$. Taking limit $k \to \infty$, we get

$$\lim_{k\to\infty} \|y_k^\delta - \mathcal{F}(z^k)\| = 0.$$

Hence, $y_k^\delta - \mathcal{F}(z^k) = 0$ which implies,

$$\mathcal{F}(z^k) = y_k^\delta.$$

Thus, $\mathcal{F}(\bar{z}) = y^\dagger$, which implies that $\bar{z}$ belongs to the solution set $\mathcal{S}(x^\dagger, \rho)$. Not that

$$\begin{aligned}\|y_k^\delta - \mathcal{F}(x^k)\| &= \|y_k^\delta - \mathcal{F}(z^k) + \mathcal{F}(z^k) - \mathcal{F}(x^k)\| \\ &\leq \|y_k^\delta - \mathcal{F}(z^k)\| + \|\mathcal{F}(x^k) - \mathcal{F}(z^k)\|.\end{aligned}$$

Applying the Proposition 2.2, we obtain:

$$\begin{aligned}\|y_k^\delta - \mathcal{F}(x^k)\| &\leq \|y_k^\delta - \mathcal{F}(z^k)\| + \frac{1}{1-\gamma}\|\mathcal{F}'(z^0)(x^k - z^k)\|, \\ &\leq \|y_k^\delta - \mathcal{F}(z^k)\| + \frac{1}{1-\gamma}\|(z^k - x^k)\|, \\ &\leq \|y_k^\delta - \mathcal{F}(z^k)\| + \frac{1}{1-\gamma}\|x^k + \kappa^k(x^k - x^{k-1}) - x^k\|, \\ &\leq \|y_k^\delta - \mathcal{F}(z^k)\| + \frac{1}{1-\gamma}\|\kappa_k(x^k - x^{k-1})\|,\end{aligned}$$

from equation (2.23), we have

$$\kappa^k(\kappa^k + 1)\|x^k - x^{k-1}\|^2 \leq \frac{\xi\|y^\delta - \mathcal{F}(z^k)\|^2}{\vartheta}.$$

Equivalently, we can write

$$(\kappa^k)^2\|x^k - x^{k-1}\|^2 \leq \frac{\xi\|y^\delta - \mathcal{F}(z^k)\|^2}{\vartheta} - \kappa^k\|x_k - x^{k-1}\|^2.$$

Thus,

$$\kappa^k\|x^k - x^{k-1}\| \leq \sqrt{\frac{\xi}{\vartheta}}\|y^\delta - \mathcal{F}(z^k)\|.$$

Note that,

$$\|y_k^\delta - \mathcal{F}(x^k)\| \leq \|y_k^\delta - \mathcal{F}(z^k)\| + \frac{1}{1-\gamma}\sqrt{\frac{\xi}{\vartheta}}\|y^\delta - \mathcal{F}(z^k)\|.$$

Taking limit $k \to \infty$, we get

$$\lim_{k\to\infty} \|y_k^\delta - \mathcal{F}(x^k)\| \leq \lim_{k\to\infty} \|y_k^\delta - \mathcal{F}(z^k)\| + \frac{1}{1-\gamma}\sqrt{\frac{\xi}{\vartheta}} \lim_{k\to\infty} \|y^\delta - \mathcal{F}(z^k)\|. \tag{2.54}$$

Hence,

$$\lim_{k\to\infty} \|y_k^\delta - \mathcal{F}(z^k)\| = 0.$$

Using it in (2.54) we get, $\mathcal{F}(\bar{x}) = y^\dagger$, which means that $\bar{x}$ belongs to the solution set $\mathcal{S}(x^\dagger, \rho)$. This Complete the proof. □

Now we show our method is asymptotically stable. For this first we get definition of it.

Definition 2.1. ([3]) Let $\delta > 0$ be given and $\{x_n^\delta\}$ be a sequence which is defined iteratively by an iteration method and $\{\delta_k\}$ be a sequence of real numbers which convergens to zero. The method is said to be asymptotically stable if there exist a subsequence $\{\delta_{k_i}\}$ of $\{\delta_k\}$ and for $\bar{n} := \lim_{i\to\infty} \hat{n}_{k_i} \in \mathbb{N} \cup \{\infty\}$, following two condition holds.

1. For all $0 \le n \le \bar{n}$
$$\lim_{i\to\infty} \{x_n^{\delta_{k_i}}\} = \tilde{x}_n, \tag{2.55}$$
for some $\tilde{x}_n \in \bar{\mathcal{B}}_\rho(x_0)$

2. If $\bar{n} = \infty$ then a solution $\tilde{x} \in \mathcal{S}(x_0, \rho)$ such that:
$$\lim_{i\to\infty} \tilde{x}_n = \tilde{x}$$

Lemma 2.14. Let sequences $\{x_n^\delta\}$ and $\{z_n^\delta\}$ be assin (2.10) and (2.13). Let $\{\delta_{k_i}\}$ be the subsequence of $\{\delta_k\}$ and $\{x_n^\delta\} \subset \bar{\mathcal{B}}_\rho(x_0)$ such that $x_n^{\delta_{k_i}} \to \tilde{x}_n$ as $i \to \infty$. Then for,
$$\tilde{z}_n = \tilde{x}_n + \kappa_n(\tilde{x}_n - \tilde{x}_{n-1}),$$
we have, $z_n^{\delta_{k_i}} \to \tilde{z}_n \quad as \quad i \to \infty$, for every $0 \le n \le \bar{n}$, where $\bar{n} := \lim_{i\to\infty} \hat{n}_{k_i} \in \mathbb{N} \cup \{\infty\}$.

Proof. The proof of result is similar to one given in [[3], Lemma 4.11], so it is omitted, here. □

Lemma 2.15. Let Assumption 2.1 be satisfied. Let $\{\delta_{k_i}\}$ be the subsequence of $\{\delta_k\}$ and $\{\tilde{x}_n\},\ \{\tilde{z}_n\} \in \bar{\mathcal{B}}_\rho(x_0)$. Then condition (1) of Definition-2.1 holds.

Proof. we can conseder two cases: (i) Case(1):- when $\bar{n} < \infty$, and (ii) Case(2):- when $\bar{n} = \infty$; we prove it for the Case(2) and using similar technique one can prove it for the Case(1) also. Using induction hypothesis we show there exist $\tilde{x}_n,\ \tilde{z}_n \subset \bar{\mathcal{B}}_\rho(x_0)$ and a subsequence $\{\delta_{k_i}\}$ that satisfy (2.55).
For $n = 0$, the sequence (2.55) implies that $\tilde{x}_0 = x_0 \subset \bar{\mathcal{B}}_\rho(x_0)$. Now, we assume that conclusion hold for some n, i.e. there exist a subsequence $\{\delta_{k_i}\}$ satisfying
$$x_n^i \to \tilde{x}_n \quad \text{as} \quad i \to \infty \tag{2.56}$$
for some $\tilde{x}_n \in \bar{\mathcal{B}}_\rho(x_0)$ and from Lemma 2.14, we have $z_n^i \to \tilde{z}_n \quad \text{as} \quad i \to \infty$. Now, we show the conclusion holds for $n + 1$. We define
$$u_n^i := \mathcal{F}'(z_0^i)^* \nu_n^i$$
$$\sigma_n^i := \min\left\{ \frac{\sigma_0 \langle \nu_n^i, r_n^i \rangle}{\|u_n^i\|^2}, M \right\},$$
where $r_n^i := \mathcal{F}(z_n^i) - y^i, \nu_n^i := (\mathcal{F}'(z_0^i)\mathcal{F}'(z_0^i)^* + \beta_n^i I)^{-1} r_n^i$. Note that $\{\sigma_n^i\}$ is bounded in Z, and $Z \hookrightarrow X$ is compactly immbeded. So, there exist $\{\sigma_n^i\} \in Z$ and a subsequence of $\{\delta_{k_i}\}$ such that
$$u_n^i \to \tilde{u}_n \quad as \quad i \to \infty. \tag{2.57}$$
Also, due to boundedness of $\{\|\mathcal{F}'(z_0^i)\|\}, \{\|\mathcal{F}'(z_0^i)^*\|\}, \{\beta_n^i\}$ being bounded, there exist r_n^i, ν_n^i, and a subsequence $\{\delta_{k_i}\}$ which satisfy.
$$r_n^i \to \tilde{r}_n, \quad \nu_n^i \to \tilde{\nu}_n \quad as \quad i \to \infty. \tag{2.58}$$

Thus, from (2.57) and (2.58) we have

$$\sigma_n^i \to \tilde{\sigma}_n \quad as \quad i \to \infty. \tag{2.59}$$

Since $x_{n+1}^i = z_n^i - \sigma_n^i u_i^n$, we obtain:

$$\tilde{x}_{n+1} = \tilde{z}_n - \hat{\sigma}_n \tilde{u}_n \quad as \quad i \to \infty,$$

where

$$\hat{\sigma} = \begin{cases} \tilde{\sigma}_n, & \text{if} \quad \mathcal{F}(\tilde{z}_n) \neq y \\ 0, & \text{if} \quad \mathcal{F}(\tilde{z}_n) = y. \end{cases}$$

Thus (2.55) holds for $n = n+1$, if $\tilde{x}_{n+1} = \tilde{z}_n - \hat{\sigma}_n \tilde{u}_n$, we get, $\tilde{x}_{n+1} \in \bar{\mathcal{B}}_\rho(x_0)$. A similar conclusion can be derived for sequence $\{\tilde{z}_n\}$.

□

The follows Lemma, we show our method satisfies condition (2) of Definition 2.1.

Lemma 2.16. Let conditions of Lemma 2.15 be satisfied. Then from sequences $\{\tilde{x}_n\}$ and $\{\tilde{z}_n\}$ converge strongly to $\bar{x}$, where $\bar{x} \in \bar{\mathcal{B}}_\rho(x_0)$, as $n \to \infty$.

Proof. From Proposition 2.6, (2.57) and (2.59), we get

$$\begin{aligned} \|\tilde{x}_{n+1} - x^*\|^2 &= \lim_{i\to\infty} \|x_{n+1}^i - x^*\|^2 \\ &\leq \lim_{i\to\infty} (\|z_n^i - x^*\| - \|\xi r_n^i\|^2) \\ &\leq \left(\|\tilde{z}_n - x^*\|^2 - \xi\|\tilde{r}_n\|^2\right), \end{aligned}$$

which implies

$$\|\tilde{x}_{n+1} - x^*\|^2 \leq \|\tilde{z}_n - x^*\|^2 - \xi\|\mathcal{F}(\tilde{z}_n) - y\|^2. \tag{2.60}$$

From Lemma 2.8 and condition (2.23) we get

$$\|\tilde{x}_{n+1} - x^*\|^2 \leq \|\tilde{x}_n - x^*\|^2$$

and $\tilde{x} \in \bar{\mathcal{B}}_\rho(x_0)$ for all $0 \leq n \leq \bar{n}$.
Note that

$$\sum_{n=0}^{\infty} \|\mathcal{F}(\tilde{z}_n) - y\|^2 < \infty, \quad \sum_{n=0}^{\infty} \kappa_n \|\tilde{x}_n - \tilde{x}_{n-1}\|^2 < \infty. \tag{2.61}$$

Using similar technique as in Theorem 2.12, it can be shown that $\{\tilde{z}_n\}$ is Cauchy sequence. Consequently, there exists a point $\tilde{x} \in \bar{\mathcal{B}}_\rho(x_0)$ such that $\tilde{x}_n \to \bar{x}$. Using continuety of $\mathcal{F}$ and the fact thst $\mathcal{F}(\tilde{x}_n) \to \mathcal{F}(\tilde{x})$, as $n \to \infty$. We therefore get, $\bar{x} \in \mathcal{S}(x_0, \rho)$. □

2.8 Main result

Now we give our main convergence result whose proof techniques are similar to the one considered in ([1]).

Theorem 2.17. Let Assumption 2.1 be hold. Then any subsequence of $\{x_{\hat{n}_k}^{\delta_k}\}$ contains a subsequence that converges strongly to an element in $\mathcal{S}(x_0, \rho)$. Moreover, if $x^\dagger$ is the unique solution of (1.1) then $x_{\hat{n}_k}^{\delta_k} \to x^\dagger$ in X as $k \to \infty$.

Proof. By Lemma 2.15, there exists a sequence $\{\tilde{x}_n\} \in \bar{\mathcal{B}}(x_0, \rho)$, satisfying condition (1) and (2) of Definition 2.1. First, for some $\hat{n} \in \mathbb{N}$, we assume that $\lim_{i\to\infty} \hat{n}_{k_i} = \tilde{n}$. From condition (1) of Definition-2.1 , we have

$$x_{\hat{n}}^{\delta_{k_i}} \to \tilde{x}_{\hat{n}} \quad as \quad i \to \infty$$

Also, from stopping rule (1.5), we have

$$\|\mathcal{F}(x_{\hat{n}}^{\delta_{k_i}}) - y^{\delta_{k_i}}\|^2 \leq \tau\delta_{k_i}$$

Taking limit $i \to \infty$, we get

$$\lim_{i\to\infty} \|\mathcal{F}(x_{\hat{n}}^{\delta_{k_i}}) - y^{\delta_{k_i}}\|^2 \leq \lim_{i\to\infty} \tau\delta_{k_i},$$

implies that $\delta_{k_i} \to 0$ as $i \to \infty$, we have

$$\|\mathcal{F}(\tilde{x}_{\hat{n}}) - y\|^2 = 0$$

Thus,

$$\mathcal{F}(\tilde{x}_{\hat{n}}) = y.$$

Thus $\tilde{x}_{\hat{n}} \in \mathcal{S}(x_0, \rho)$.
Now we assume $\hat{n}_{k_i} \to \infty$ as $i \to \infty$. From condition (2) of Definition-2.1 there exist $\tilde{x}_n, \tilde{x} \in \mathcal{S}(x_0, \rho)$ and subsequence $\{\delta_{k_i}\}$ of $\{\delta_k\}$ such that $\lim_{i\to\infty}\{x_n^{\delta_{k_i}}\} = \tilde{x}_n$, for $0 \leq n \leq \tilde{n}_{k_i}$, for i large enough and $\lim_{n\to\infty} \tilde{x}_n = \tilde{x}$. Thus, for a given $\epsilon > 0$, there exists an integer k_* such that

$$\|\tilde{x}_{k_*} - \tilde{x}\| < \frac{\epsilon}{2}.$$

Note that $\hat{n}_{k_i} \to \infty$ as $i \to \infty$ and there exist $\bar{i} \in \mathbb{N}$ such that

$$k_* \leq \hat{n}_{k_i} \quad and \quad \|\tilde{x}_{k_*}^{\delta_{k_i}} - \tilde{x}\| < \frac{\epsilon}{2},$$

for all $i \geq \bar{i}$. Thus, by Proposition 2.7 and traingle inequality, we get

$$\|x_{\hat{n}_{k_i}}^{\delta_{k_i}} - \tilde{x}\| \leq \|x_{k_*}^{\delta_{k_i}} - \tilde{x}\| \leq \|x_{k_*}^{\delta_{k_i}} - \tilde{x}_{k_*}\| + \|\tilde{x}_{k_*} - \tilde{x}\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2},$$

which implies,

$$\|x_{\hat{n}_{k_i}}^{\delta_{k_i}} - \tilde{x}\| \leq \epsilon$$

for all $i \geq \bar{i}$. Thus, we obtain that $\lim_{i\to\infty} \|x_{\hat{n}_{k_i}}^{\delta_{k_i}} - \tilde{x}\| = 0$, which implies that $x_{\hat{n}_{k_i}}^{\delta_{k_i}} \to \tilde{x}$ as $i \to \infty$. □

3 Conclusion

In this dissertation, we have considered a modified version of two-point Levenberg-Marquardt method to find stable approximate solution of the nonlinear ill-posed operator equation (1.1). The Modified version of method considered in [3] has a advantage that it requires calculation of Fréchet derivate only at an initial guess x_0. In the dissertation, we have done convergence analysis method with and without data error

4 BIBLIOGRAPHY

References

[1] C. Clason, V. H. Nhu, "Bouligand-Landweber iteration for a non-smooth ill-posed problem," Numerische Mathematik 142(4), (2019), Volume 142, 789-832.

[2] H.W. Engl, M. Hanke and A. Neubauer, "Regularization of inverse problems," Kluwer Academic Publishers, Dordrecht, 1996.

[3] G. Gao et.al.,"A genralized adaptive Levenberg-Marquardt method for solving nonlinear ill-posed problems," Mathematics of computation, (2022), 39(2023), 015009(29pp).

[4] C. W. Groetsch, "The theory of Tikhonov regularization for Fredholm equations of the first kind," Research Notes in Mathematics, vol. 105, Pitman (Advanced Publishing Program),Boston, MA, 1984.

[5] M. Hanke,"A regularizing Levenberg-Marquardt scheme with applications to inverse groundwater filetration problems," Inverse Problems 13 (1997), no. 1, 7995.

[6] M. Hanke, A. Neubauer and O. Scherzer, "A convergence analysis of Landweber iteration for nonlinear ill-posed problems," Numer. Math. 72 (1995), 2137.

[7] S. Hubmer and R. Ramlau , "Nesterovs accelerated gradient method for nonlinear ill-posed problems with a locally convex residual functional," Inverse Problems, 2018, no. 34 095003.

[8] Q. Jin, " On a regularized LevenbergMarquardt method for solving nonlinear inverse problems," 2010, Numer.Math.,(2010), no.115,229-59.

[9] B. Kaltenbacher, A. Neubauer and O. Scherzer, "Iterative regularization methods for nonlinear ill-posed problems," Radon Series on Computational and Applied Mathematics, vol.6, Walter de Grutyer GmbH Co. KG, Berlin, 2008.

[10] M.T. Nair, "Functional Analysis a First Course," PHI Learning Private Limited, (2014)110-092.

[11] F. Zhenwu, C. Yong , L. Li and H.Bo, "Analysis of a genralized regularized Gauss-Newton method under heuristic rule in Banach spaces," Inverse Problems, Volume 37, no. 12 (2021).